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ON THE ROLE OF CONSTANT-STRESS SURFACES IN THE PROBLEM OF MINIMIZING ELASTIC STRESS CONCENTRATION

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Abstract—This paper is concerned with conditions under which surfaces of constant stress magnitude serve as optimal from the standpoint of minimizing stress. Such conditions are established for elastic solids in the cases of antiplane shear deformation, axisymmetric torsional deformation, and plane deformation.

1. INTRODUCTION

Efforts to determine the best shape to give an elastic solid so as to minimize stress concentration often lead to inverse problems in the theory of elasticity. Ordinarily such problems consist of prescribing the forces and their place of application, and seeking the shape of a portion of the free surface in such a way as to minimize the maximum stress acting there. This unknown part of the boundary might be associated with a hole, fillet, or notch.

In the context of the plane theory, interesting problems have been treated by Neuber[1, 2] and Cherepanov[3]. These investigations involved as a hypothesis the assumption that the best shape is one that gives rise to stress of constant magnitude over the unknown part of the boundary. This intuitively compelling assumption greatly simplifies the task of finding optimal shapes, and clearly applies to a far wider class of technically important problems than those considered in [1–3]. Although it is doubtful whether the origin of this idea can be documented, it is worth mentioning that it was considered as long ago as 1934 by R. V. Baud[4]. Another impetus for this hypothesis stems from the fact that in certain circumstances it is associated with a surface of uniform yielding.

The purpose of the present investigation is to discuss a number of instances involving antiplane shear deformation, axisymmetric torsion, and the plane strain theory, where a rigorous verification of the optimal nature of constant-stress surfaces is within reach. The results for the plane theory, which pertain to exterior, doubly-connected domains are motivated in part by a remark of Cherepanov[3, p. 929] to the effect that a mathematical confirmation appears to be lacking. In this paper, Cherepanov studied the problem for exterior domains of arbitrary connectivity, and came up with the remarkable fact that in the case of a single hole, a certain ellipse describes a free surface of constant stress magnitude.

In the case of antiplane shear deformation, a detailed analysis does not seem warranted. Rather, it is sufficient to call attention to the analogy with plane potential flow, and to point out that free streamlines interpret as the profiles of traction free surfaces of constant stress magnitude. A result due to Garabedian and Spencer[5] furnishes conditions under which such streamlines bear the least maximum velocity compared with other streamlines. Because of the analogy, such streamlines serve as profiles of least maximum stress magnitude. This theorem was discussed in clear terms by Gilbarg[6, Section 31], who refers to it as the Minimax Principle. Numerous two-dimensional free-streamline problems are analyzed in [6–8], and many of them have interesting interpretations as optimal shape problems associated with antiplane shear deformations. In particular, the Riabouchinski cavity problem is related to the problem of profiling the tip of a groove (notch) in an elastic half-space.

Although the Minimax Principle encompasses axisymmetric flows, the analogy between such flows and axisymmetric torsion lacks the conclusiveness possessed by the one between plane flow and anti-plane shear. We have therefore taken up the torsion problem in detail in Section 2. The discussion in Section 31 of [6] served as a valuable guide in this analysis, and there is no major departure from the key steps suggested by this discussion.

2. AXISYMMETRIC TORSION

We are concerned with the stresses arising from the torsion of an elastic solid lying within a radially convex region of revolution $\mathcal{R}$ having plane ends. Let (r, θ, z) denote cylindrical coordinates such that the z-axis coincides with the axis of $\mathcal{R}$, and let (x_1, x_2, x_3) stand for the rectangular Cartesian coordinates related by

$$\left.\begin{array}{l} x_1 = z, \quad x_2 = r\cos\theta, \quad x_3 = r\sin\theta \\ 0 \le r < \infty, \quad 0 \le \theta < 2\pi, \quad -\infty < z < \infty. \end{array}\right\} \tag{2.1}$$

Body force is assumed to be absent, and the lateral surface to be traction free. The boundary conditions for the ends, Π_α $(\alpha = 1, 2)$,† have the form

$$\tau_{zr} = \tau_{zz} = 0, \quad \tau_{z\theta} = t^{(\alpha)}(r) \quad \text{on } \Pi_\alpha, \tag{2.2}$$

where τ_{zr}, τ_{zz}, $\tau_{z\theta}$ are cylindrical components of stress and $t^{(\alpha)}$ are prescribed functions.

In Fig. 1, $\mathcal{M}$ stands for the open meridional section corresponding to $\theta = 0$, whereas Γ indicates the intersection of the lateral surface with the plane $\theta = 0$ and L_α the intersections formed by this plane and the ends Π_α.

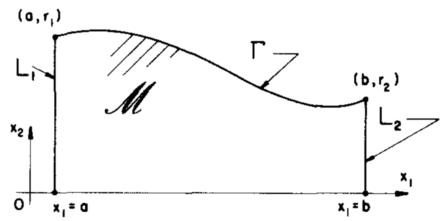

Fig. 1.

Because we are interested in the state of stress, it is convenient to introduce the Michell torsion function.‡ This function, which we here denote by ψ, is assumed to be twice continuously differentiable on $\mathcal{M}$ and once so on its closure, i.e.

$$\psi \in \mathscr{C}^1(\bar{\mathcal{M}}) \cap \mathscr{C}^2(\mathcal{M}). \tag{2.3}$$

The boundary-value problem for ψ consists of the partial differential equation

$$\psi_{,11} + \psi_{,22} - \frac{3}{x_2}\psi_{,2} = 0 \quad \text{on } \mathcal{M}, \tag{2.4§}$$

and the boundary conditions

$$\left.\begin{array}{ll} \psi(a, x_2) = m^{(1)}(x_2) & \text{for } 0 \le x_2 \le r_1, \\ \psi(b, x_2) = m^{(2)}(x_2) & \text{for } 0 \le x_2 \le r_2, \end{array}\right. \tag{2.5}$$

$$\psi(x_1, 0) = 0 \quad \text{for } a \le x_1 \le b, \qquad \psi = \frac{M}{2\pi} \quad \text{on } \Gamma, \tag{2.6}$$

where

$$m^{(\alpha)}(x_2) = \int_0^{x_2} \rho^2 t^{(\alpha)}(\rho)\, d\rho, \quad M = m^{(1)}(r_1) = m^{(2)}(r_2). \tag{2.7}$$

The last of (2.7) expresses equilibrium of the applied forces, M being the resultant moment. We

†We adopt the convention that Greek letter indices have the range $\{1, 2\}$.
‡For a discussion of this function, see [9, Section 49].
§The subscripts preceded by a comma indicate partial differentiation.

assume that M is positive. As for the state of stress, the cylindrical components are given by

$$\tau_{zz} = \tau_{rr} = \tau_{\theta\theta} = \tau_{rz} = 0 \quad \text{on } \mathcal{R}, \tag{2.8}$$

$$\begin{aligned} \tau_{r\theta} &= \tau_{r\theta}(r, 0, z) = \tau_{r\theta}(x_2, 0, x_1) = -\frac{1}{x_2^2}\psi_{,1} \\ \tau_{z\theta} &= \tau_{z\theta}(r, 0, z) = \tau_{z\theta}(x_2, 0, x_1) = \frac{1}{x_2^2}\psi_{,2}. \end{aligned} \tag{2.9}$$

This justifies calling τ defined by

$$\tau = (\tau_{r\theta}^2 + \tau_{z\theta}^2)^{1/2} \quad \text{on } \bar{\mathcal{M}}. \tag{2.10}$$

the *stress magnitude.* By (2.9),

$$\tau = \frac{1}{x_2^2}(\psi_{,1}{}^2 + \psi_{,2}{}^2)^{1/2} \quad \text{on } \bar{\mathcal{M}}, \tag{2.11}$$

which in view of (2.6), furnishes the following relation between τ and the outward normal derivative of ψ along Γ:

$$\tau = \frac{1}{x_2^2}\left|\frac{\partial\psi}{\partial n}\right|$$

at points of Γ where the outward unit normal vector is uniquely defined. We assume that Γ has a piecewise continuous normal vector n and that Γ is not reentrant at points where n is discontinuous. We shall refer to such curves as *admissible,* and assume

$$\tau(x) = 0$$

if $x = (x_1, x_2)$ is a point of Γ at which the normal is discontinuous.

The following version of the Maximum Principle for (2.4) is important to our analysis:

Theorem 1. Let ψ satisfy (2.3), (2.4), and assume

$$\psi(x_1, 0) = 0 \quad \text{for } a \le x_1 \le b. \tag{2.13}$$

Then, for every $x = (x_1, x_2) \in \mathcal{M}$,

$$\inf_{\partial\mathcal{M}} \psi \le \psi(x) \le \sup_{\partial\mathcal{M}} \psi, \tag{2.14}$$

where $\partial\mathcal{M}$ denotes the boundary of $\mathcal{M}$.

A detailed proof would no doubt be out of place, but a few words are in order since eqn (2.4), though elliptic, fails to posses bounded coefficients. Let $\delta_0 > 0$ be such that

$$\mathcal{M}_\delta = \mathcal{M} - \{(x_1, x_2) | a \le x_1 \le b, 0 \le x_2 \le \delta\} \tag{2.15}$$

is connected for every $\delta \in (0, \delta_0)$. For each such δ, (2.4) is uniformly elliptic on $\mathcal{M}_\delta$ and there has continuous, bounded coefficients. Consequently, the Maximum Principle as given in [10] furnishes

$$\inf_{\partial\mathcal{M}_\delta} \psi \le \psi(x) \le \sup_{\partial\mathcal{M}_\delta} \psi$$

for every $x \in \bar{\mathcal{M}}_\delta$. It is now an elementary matter to show that (2.3) and (2.13) imply (2.14).

With the aid of Theorem 1, we can establish the following useful result.

Theorem 2. Suppose that $\bar{\psi}$, $\hat{\psi}$ satisfy (2.3), (2.4) and (2.13). Assume

$$\hat{\psi} \geq \bar{\psi} \quad \text{on } \Gamma \cup L_1 \cup L_2, \tag{2.16}$$

and let $\mathring{x} \in \Gamma \cup L_1 \cup L_2$ be such that

$$\hat{\psi}(\mathring{x}) = \bar{\psi}(\mathring{x}). \tag{2.17}$$

Then,

$$\frac{\partial \bar{\psi}}{\partial n} \geq \frac{\partial \hat{\psi}}{\partial n} \quad \text{at } \mathring{x}. \qquad (2.18)\dagger$$

Proof. Let $\psi = \hat{\psi} - \bar{\psi}$, and apply (2.16), (2.17), and Theorem 1 to conclude that

$$\psi \geq 0 \quad \text{on } \mathcal{M}, \qquad \psi(\mathring{x}) = 0.$$

Accordingly, and since $\psi \in \mathscr{C}^1(\bar{\mathcal{M}})$, (2.18) follows.

Our next task is to establish that solutions of (2.4) remain solutions under a homogeneous change of coordinates. Let $k > 0$ and define

$$x'_\alpha = kx_\alpha. \tag{2.19}$$

According as $k \in (0, 1)$, $k = 1$, or $k > 1$, we have contraction toward the origin O, the identity map, or a dilation with respect to O.

Let ψ satisfy (2.3), (2.4), and define ψ' on

$$\mathcal{M}' = \{x' | x' = kx, \quad x \in \mathcal{M}\} \tag{2.20}$$

through

$$\psi'(x') = \psi(x) = \psi(x'/k). \tag{2.21}$$

Then,

$$\psi'_{,\alpha}(x') = \frac{1}{k}\psi_{,\alpha}(x), \quad \psi'_{,\alpha\beta}(x') = \frac{1}{k^2}\psi_{,\alpha\beta}(x). \qquad (2.22)\ddagger$$

Therefore, and by (2.19), ψ' satisfies (2.4), i.e.

$$\psi'_{,11}(x') + \psi'_{,22}(x') - \frac{3}{x'_2}\psi'_{,2}(x') = 0 \tag{2.23}$$

for all $x' \in \mathcal{M}'$. We have thus established.

Theorem 3. Let ψ satisfy (2.3), (2.4), and define ψ' on $\mathcal{M}'$ through (2.19), (2.21), where $k > 0$ and $\mathcal{M}'$ is given by (2.20). Then

$$\psi' \in \mathscr{C}^1(\bar{\mathcal{M}}) \cap \mathscr{C}^2(\mathcal{M}),$$

and ψ' satisfies (2.23) on $\mathcal{M}'$.

We turn now to the derivation of a result which is useful in comparing among values of τ on Γ as Γ is allowed to vary. Let $\mathcal{M}$ and $\mathcal{M}^*$ be two radially convex meridional domains such that $L_\alpha = L^*_\alpha$. Assume that O and $k \in (0, 1]$ can be chosen such that $\mathcal{M}' \subset \mathcal{M}^*$, where $\mathcal{M}'$ is given by

†This conclusion remains valid when the derivatives are interpreted as directional derivatives in an outward direction, an interpretation which is essential if the normal vector lacks continuity at $\mathring{x}$.

‡Here and in the sequel, subscripts preceded by a comma refer to partial differentiation with respect to the corresponding independent variable.

(2.20), and such that Γ', Γ^* have in common a point $\mathring{x}$ (see Fig. 2). We assume that Γ, Γ^* are admissible curves. It follows that Γ' is also admissible.

Let ψ satisfy (2.3)–(2.6) and let ψ^* be such that

$$\psi^* \in \mathscr{C}^1(\bar{\mathscr{M}}^*) \cap \mathscr{C}^2(\mathscr{M}^*),$$

$$\psi^*_{,11} + \psi^*_{,22} - \frac{3}{x_2}\psi^*_{,2} = 0 \quad \text{on } \mathscr{M}^*, \tag{2.24}$$

$$\begin{aligned} \psi^*(a, x_2) &= m^{(1)}(x_2) \quad \text{for } 0 \le x_2 \le r_1, \\ \psi^*(b, x_2) &= m^{(2)}(x_2) \quad \text{for } 0 \le x_2 \le r_2, \end{aligned} \tag{2.25}$$

$$\psi^*(x_1, 0) = 0 \quad \text{for } a \le x_1 \le b, \quad \psi^* = \frac{M}{2\pi} \quad \text{on } \Gamma^*. \tag{2.26}$$

Our objective is to establish the inequality

$$\tau^*(\mathring{x}) \ge \tau(\mathring{x}/k). \tag{2.27}$$

Since we are assuming that τ^*, τ vanish at points of discontinuity of the normal to Γ^*, Γ, we may without loss of generality take $\mathring{x}$ to be a point at which n', n^* are continuous.

Define

$$\hat{\psi} = \psi' - \psi^* \quad \text{on } \mathscr{M}'. \tag{2.28}$$

By (2.4), (2.24), and Theorem 3,

$$\hat{\psi}_{,11} + \hat{\psi}_{,22} - \frac{3}{x_2}\hat{\psi}_{,2} = 0 \quad \text{on } \mathscr{M}'. \tag{2.29}$$

At points of L'_α, we find from (2.5), (2.21),

$$\psi'(a', x_2') = m^{(1)}(x_2'/k), \quad \psi'(b', x_2') = m^{(2)}(x_2'/k). \tag{2.30}$$

Assume $m^{(\alpha)}$ are non-decreasing functions. Then (2.25), (2.28), (2.30) furnish

$$\hat{\psi} \ge 0 \quad \text{on } L'_\alpha, \tag{2.31}$$

since $k \in (0, 1)$. From (2.6) and (2.21) there follows

$$\psi' = \frac{M}{2\pi} \quad \text{on } \Gamma'. \tag{2.32}$$

Since $m^{(\alpha)}$ are monotone functions, (2.25), (2.26) and Theorem 1 imply

$$\psi^* \le \frac{M}{2\pi} \quad \text{on } \Gamma'. \tag{2.33}$$

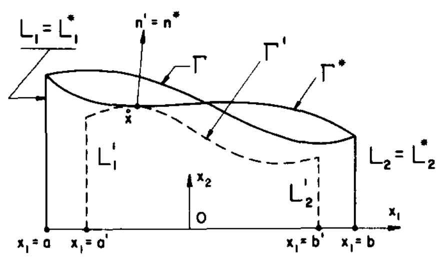

Fig. 2.

Accordingly, and since $\mathring{x} \in \Gamma' \cap \Gamma^*$, (2.26), (2.28) and (2.32) imply

$$\hat{\psi} \geq 0 \quad \text{on } \Gamma', \qquad \hat{\psi}(\mathring{x}) = 0. \tag{2.34}$$

Next, it is clear from (2.6), (2.21), (2.26) and (2.28) that

$$\hat{\psi}(x_1, 0) = 0, \quad a' \leq x_1 \leq b'. \tag{2.35}$$

In view of (2.28), (2.29), (2.31) and (2.34), Theorem 2 thus yields

$$\frac{\partial \psi'}{\partial n} \leq \frac{\partial \psi^*}{\partial n} \quad \text{at } \mathring{x}. \tag{2.36}$$

Similarly, Theorem 2, together with (2.23), (2.30), (2.32), and the fact that $\psi'(x_1, 0) = 0$ for $a' \leq x_1 \leq b'$ imply

$$\frac{\partial \psi'}{\partial n} \geq 0 \quad \text{on } \Gamma'. \tag{2.37}$$

Combining (2.36), (2.37), we see that

$$0 \leq \frac{\partial \psi'}{\partial n} \leq \frac{\partial \psi^*}{\partial n} \quad \text{at } \mathring{x}. \tag{2.38}$$

Since

$$\tau^* = \frac{1}{x_2^2}(\psi^{*2}{}_{,1} + \psi^{*2}{}_{,2})^{1/2} \quad \text{on } \bar{\mathcal{M}}^*,$$

and

$$\tau' = \frac{1}{x_2^2}(\psi'^2{}_{,1} + \psi'^2{}_{,2})^{1/2} \quad \text{on } \bar{\mathcal{M}}', \tag{2.40}$$

and since ψ', ψ^* are constant on Γ', Γ^*, we have at $\mathring{x}$

$$\tau^* = \frac{1}{\mathring{x}_2^2}\left|\frac{\partial \psi^*}{\partial n}\right|, \quad \tau' = \frac{1}{\mathring{x}_2^2}\left|\frac{\partial \psi'}{\partial n}\right|. \tag{2.41}$$

Consequently, (2.38) implies

$$\tau^*(\mathring{x}) \geq \tau'(\mathring{x}). \tag{2.42}$$

In order to complete the argument, we need to relate $\tau'(\mathring{x})$ and $\tau(\mathring{x}/k)$. From (2.40) and since

$$\psi'_{,\alpha}(x) = \frac{1}{k}\psi_{,\alpha}(x/k), \tag{2.43}$$

it follows that

$$\tau'(x) = \frac{1}{k^3}\left\{\frac{1}{(x_2/k)^2}[\psi^2{}_{,1}(x/k) + \psi^2{}_{,2}(x/k)]^{1/2}\right\}. \tag{2.44}$$

Hence, (2.11) furnishes

$$\tau'(x) = \frac{1}{k^3}\tau(x/k). \tag{2.45}$$

The desired conclusion (2.27) follows from (2.42) and the assumption $k \in (0, 1]$.

We summarize the results of the foregoing analysis in:

Theorem 4. Let $\mathcal{M}$ and $\mathcal{M}^*$ be radially convex meridional domains for which $L_\alpha = L_\alpha^*$ and Γ, Γ^* are admissible. Assume there exists $k \in (0, 1]$ such that $\mathcal{M}'$ given by (2.20) is contained in $\mathcal{M}^*$ and Γ', Γ^* have a common point $\mathring{x}$. Assume that ψ satisfies (2.3)–(2.6) and that ψ^* satisfies (2.23)–(2.26), with $m^{(\alpha)}$ nondecreasing functions. Finally, let

$$\tau = \frac{1}{x_2^2}(\psi^2{}_{,1} + \psi^2{}_{,2})^{1/2} \quad \text{on } \mathcal{M}, \qquad \tau^* = \frac{1}{x_2^2}(\psi^{*2}{}_{,1} + \psi^{*2}{}_{,2})^{1/2} \quad \text{on } \mathcal{M}^*,$$

and assume that τ, τ^* vanish at points of Γ, Γ^* where the normal is discontinuous. Then,

$$\tau^*(\mathring{x}) \geq \tau(\mathring{x}/k). \tag{2.46}$$

We come now to our main purpose, the application of the foregoing theorems to the problem of optimal profiles. Consider a pair of domains $\mathcal{M}$ and $\mathcal{M}^*$ of the type indicated by Fig. 3, whose boundaries coincide except for parts Λ, Λ^* of Γ, Γ^*. Let ψ, ψ^* satisfy the hypothesis of Theorem 3, and assume

$$\tau = \bar{\tau} \text{ (constant) on } \Lambda. \tag{2.47}$$

We wish to conclude that

$$\sup_{\Lambda^*} \tau^* \geq \bar{\tau}. \tag{2.48}$$

There are the two cases $\Lambda \subset \mathcal{M}^*$ and $\Lambda \not\subset \mathcal{M}^*$ to consider. Concerning the first of these, we note that unless the endpoints of Λ and Λ^* are points of continuity of the normals, (2.48) is trivially true because $\bar{\tau} = 0$ in that instance. Assuming that the normals are continuous at the endpoints, it is easy to show with the aid of Theorem 2 that

$$0 \leq \frac{\partial\psi}{\partial n} \leq \frac{\partial\psi^*}{\partial n} \quad \text{at the endpoints of } \Lambda.$$

Therefore, and since ψ, ψ^* are constant on Γ, Γ^*, (2.48) follows.

In the event that $\Lambda \not\subset \mathcal{M}^*$, we assume these exists $k \in (0, 1]$ and a choice of O such that $\mathcal{M}' \subset \mathcal{M}^*$ and Λ^*, Λ' have a point $\mathring{x}$ in common. Apply Theorem 3 to conclude

$$\tau^*(\mathring{x}) \geq \bar{\tau},$$

from which (2.48) is seen to hold.

The key to the applicability of the foregoing result is the question whether a suitable contraction is possible. A reasonably accurate idea of how contraction affects a region is afforded by fact that a point and its image lie on a ray through O, and that the image of a straight line is a parallel line. In Fig. 4a, Λ profiles a fillet in a stepped shaft, and an appropriate contraction is clearly possible for the fixed part of $\partial\mathcal{M}$. The sketch in Fig. 4b corresponds to the problem in which Λ is an optimal root profile for a notch. The point P, which is the intersection of the

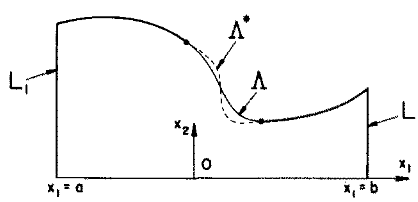

Fig. 3.

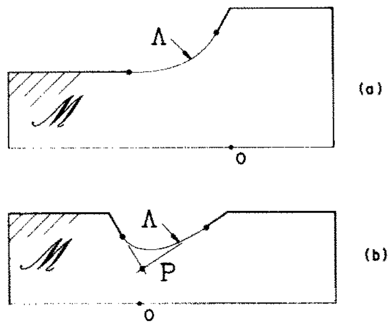

Fig. 4.

(x_1, x_2)-plane and the cones formed by the notch faces, must lie within $\mathcal{M}$ for a contraction to be possible. Unfortunately, this places beyond the scope of the present treatment the interesting case in which the faces are at right angles to the axis of the shaft.

3. EXTERIOR PROBLEMS OF TWO-DIMENSIONAL ELASTOSTATICS

Let D be a plane domain bounded internally by a single closed contour C. We are interested in the stresses arising from the application of loads at infinity in the case when C is traction free and body force is absent. For plane deformation, the stresses $\tau_{\alpha\beta}$ and displacements u_α obey the equation of equilibrium

$$\tau_{\alpha\beta,\beta} = 0, \tag{3.1}$$

and the stress-displacement relations

$$\tau_{\alpha\beta} = \lambda u_{\gamma,\gamma}\delta_{\alpha\beta} + 2\mu u_{(\alpha,\beta)}, \tag{3.2}†$$

λ being the Lamé modulus and μ the shear modulus. The condition that C be traction free is expressed as

$$\tau_{\alpha\beta}n_\beta = 0 \quad \text{on } C, \tag{3.3}$$

where n_β designate the components of the unit normal outward from D. The loading is described by the requirement that

$$\tau_{\alpha\beta} \to \overset{\infty}{\tau}_{\alpha\beta} \quad \text{as } r \to \infty, \tag{3.4}$$

where $\overset{\infty}{\tau}_{\alpha\beta}$ are given. We assume the displacements u_α are in class $\mathscr{C}^1(\bar{D}) \cap \mathscr{C}^3(D)$. The relations (3.1)–(3.4) uniquely determine $\tau_{\alpha\beta}$, if as we assume, $\mu > 0$ and $3\lambda + 2\mu > 0$. The remaining stress components are given by

$$\tau_{33} = \nu u_{\gamma,\gamma}, \qquad \tau_{3\alpha} = 0 \quad \text{on } D, \tag{3.5}$$

where $\nu = \lambda/2(\lambda + \mu)$ denotes the Poisson ratio.

Let

$$\tau = (\tau_{\alpha\beta}\tau_{\alpha\beta})^{1/2} \quad \text{on } \bar{D}, \tag{3.6}$$

and let $\mathscr{C}$ stand for the class of all external domains bounded by a single contour. Define

$$\mathring{\tau} = \inf_{D \in \mathscr{C}} \sup_{D} \tau. \tag{3.7}$$

†Summation over repeated subscripts is implied, and those enclosed in parentheses refer to the symmetric part.

We say that C is an *optimal contour* if

$$\sup_D \tau = \mathring{\tau}. \tag{3.8}$$

Assuming there exists a domain $D^* \in \mathscr{E}$ such that $\overset{*}{\tau}$ is constant on C^*, we wish to identify conditions on $\overset{\infty}{\tau}_{\alpha\beta}$ sufficient to ensure that C^* is an optimal contour.

Let

$$\Theta = \tau_{\alpha\alpha} \quad \text{on } \bar{D}. \tag{3.9}$$

Then (3.1), (3.2) require that Θ be a harmonic function on D, i.e.,

$$\Delta\Theta = 0 \quad \text{on } D, \tag{3.10}$$

where Δ stands for the Laplace operator. By (3.3), the determinant of $\tau_{\alpha\beta}$ vanishes on C, which combines with the symmetry of $\tau_{\alpha\beta}$ to yield

$$\tau_{12}^2 = \tau_{11}\tau_{22} \quad \text{on } C.$$

Accordingly, and by (3.6), (3.9), there follows

$$\tau = |\Theta| \quad \text{on } C. \tag{3.11}$$

Another important feature of Θ,

$$\Theta \to \overset{\infty}{\tau}_{\alpha\alpha} \quad \text{as } r \to \infty, \tag{3.12}$$

is readily inferred from (3.4), (3.9).

The following theorem enables us to take advantage of the properties (3.10)–(3.12) of Θ.

Theorem 5. Let $\varphi \in \mathscr{C}(\bar{D}) \cap \mathscr{C}^2(D)$, and assume

$$\Delta\varphi = 0 \quad \text{on } D, \tag{3.13}$$

$$\varphi \to a \quad \text{as } r \to \infty. \tag{3.14}$$

Then:

(a) either $|\varphi|$ exceeds $|a|$ at a point of C, or $\varphi = a$ on $\bar{D}$;
(b) if φ is constant on C, $\varphi = a$ on $\bar{D}$.

Proof. It is well known[11, p. 248] that under an inversion of coordinates, φ is taken into a harmonic function which assumes the value a at an interior point, and it is clear that if the invariant circle is contained in the complement of $\bar{D}$, then D transforms to a bounded domain. Part (a) follows from the maximum principle and Part (b) from the uniqueness of solution to the Dirichlet problem. The proof is now complete.

Let $D^* \in \mathscr{E}$ be such that τ^* is constant on C^*. Then (3.10), (3.11), (3.12) and Part (b) of Theorem 5 imply

$$\Theta^* = \overset{\infty}{\tau}_{\alpha\alpha} \quad \text{on } \bar{D}^*, \tag{3.15}$$

$$\tau^* = |\overset{\infty}{\tau}_{\alpha\alpha}| \quad \text{on } C^*. \tag{3.16}$$

By (3.1), (3.2) and (3.15),

$$\Delta\tau^*_{\alpha\beta} = 0 \quad \text{on } D^*, \tag{3.17}$$

and it follows that $(\tau^*)^2$ is subharmonic† on D^*. Accordingly,

$$\sup_{D^*} \tau^* = \max \{(\overset{\infty}{\tau}_{\alpha\beta}\overset{\infty}{\tau}_{\alpha\beta})^{1/2}, |\overset{\infty}{\tau}_{\alpha\alpha}|\}. \tag{3.18}$$

Assume

$$|\overset{\infty}{\tau}_{\alpha\alpha}| \geq (\overset{\infty}{\tau}_{\alpha\beta}\overset{\infty}{\tau}_{\alpha\beta})^{1/2}. \tag{3.19}$$

Apply Part (a) of Theorem 5 to conclude that if $D \in \mathscr{C}$ and τ is not constant on C, then

$$\sup_D \tau \geq \sup_C \tau = \sup_C |\Theta| > |\overset{\infty}{\tau}_{\alpha\alpha}|. \tag{3.20}$$

We now summarize our findings.

Theorem **6.** Assume that $\overset{\infty}{\tau}_{\alpha\beta}$ obey (3.19), and assume there exists $D^* \in \mathscr{C}$ such that τ^* is constant on C^*. Then C^* is an optimal contour and

$$\mathring{\tau} = |\overset{\infty}{\tau}_{\alpha\alpha}|. \tag{3.21}$$

If $D \in \mathscr{C}$, and τ is not constant on C, then C is not an optimal contour.

Although this theorem does not ensure uniqueness of the optimal contour, it gives conditions under which the search should initiate with the class of those bearing constant stress magnitude. In the case of isotropic stress at infinity, where

$$\overset{\infty}{\tau}_{\alpha\beta} = \sigma\delta_{\alpha\beta}, \tag{3.22}$$

it is easily seen[9, p. 292] that the value

$$\mathring{\tau} = 2|\sigma|,$$

given by (3.21), is assumed if C is a circle no matter what its radius nor where its center. It seems likely that the source of non-uniqueness can in general be eliminated by prescribing the area and centroid of the region bounded externally by C.

Finally, we mention that if a stress concentration factor k is defined as

$$k = \frac{1}{(\overset{\infty}{\tau}_{\alpha\beta}\overset{\infty}{\tau}_{\alpha\beta})^{1/2}} \sup_D \tau,$$

then we have the lower bound

$$k \geq \frac{|\overset{\infty}{\tau}_{\alpha\alpha}|}{(\overset{\infty}{\tau}_{\alpha\beta}\overset{\infty}{\tau}_{\alpha\beta})^{1/2}}.$$

provided, of course, (3.19) holds and a constant stress solution exists.

†Pólya[12] showed in a more general setting that the stress magnitude is maximum at the surface if the dilatation is an affine function of position.

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